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Journal of APPLIED MATHEMATICS AND MECHANICS

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Journal of Applied Mathematics and Mechanics 68 (2004) 933–939

MATHEMATICAL MODELLING OF TRAFFIC FLOWS ON CONTROLLED ROADS†

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(*Received* 12 *March* 2003)

A model of transient single-lane traffic flows is proposed, taking into account the basic components used to control road traffic (traffic lights and "sleeping policemen"), which is radically different from those traditionally considered in continuum mechanics. The model takes into account the main property of traffic flows, namely, the property of self-organisation, and enables the conditions required to ensure maximum carrying capacity to be described correctly both qualitatively and quantitatively, as well as the occurrence and evolution of "travelling traffic jams" on roads, as well as the effect of road traffic control units.

Unlike the first mathematical models, which describe traffic flows [1–6] and the corresponding research, generalized in the monograph [7], a model of traffic flows was proposed in [8–10] which contains not only a continuity equation but also a differential equation of the motion, and takes into account the limits on speed and acceleration of the traffic flow, the technical characteristics of the vehicles and the features of the response of a driver to a change in the road conditions. According to this model, the problem of traffic flow has no direct hydrodynamic analogy.

Below, developing this model, we take into account additional road conditions, namely, the different forward visibility distances for a driver, and the presence on the road of traffic lights and "sleeping policemen".

1. THE MODEL OF THE TRAFFIC FLOW ALONG AN ARTERIAL ROAD

Consider the unidirectional flow of vehicles along a single-lane road. An intersection with other roads and the presence of traffic lights will be taken into account by appropriate boundary conditions. We will introduce an Euler system of coordinates x along the arterial road in the direction of the traffic flow and the time t.

We will define the mean flow density $\rho(x, t)$ as the ratio of the area of the traffic lane, occupied by the vehicles to the area of the whole section of the traffic lane considered

$$\rho = \frac{S_{\text{tr}}}{S} = \frac{hnl}{hL} = \frac{nl}{L}$$

where h is the width of the traffic lane, L is the length of the controlled section of the road, l is the mean length of the vehicle, and n is the number of vehicles in the controlled section. Thus, the flow density introduced is a dimensionless quantity, which varies from zero to unity.

We will introduce the flow velocity $\upsilon(x, t)$, which can vary from zero $\upsilon_{\max}^0$ – the maximum allowed speed on the arterial road outside the systems for controlling the traffic. It follows from the definitions that the maximum density $\rho = 1$ corresponds to the situation when the vehicles are practically up against one another ("bumper to bumper"). In this case it is natural to take $\upsilon = 0$, i.e. there is a "traffic jam" on the road.

†*Prikl. Mat. Mekh.* Vol. 68, No. 6, pp. 1035–1042, 2004.

0021–8928/$—see front matter.

doi: 10.1016/j.jappmathmech.2004.11.014

By calling the quantity

$$m = \int_0^L \rho dx$$

the "mass", concentrated in a section of length L, we can write the change in mass on the arterial road. For a continuous flow of vehicles we will have the following equation of continuity

$$\partial\rho/\partial t + \partial(\rho \upsilon)/\partial x = 0 \tag{1.1}$$

We will write the equation of the dynamics of traffic flow, more exactly, the equation of the change in the mode of motion. The mode of motion of vehicles on the road is defined by the following main factors: the response of a driver to a change in the road conditions and the actions which he takes, the response of the traffic to the driver's action, and the technical characteristics of the vehicles. In developing the model of traffic dynamics we made the following main assumptions.

1. In view of the fact that it is the average traffic that is being described, and not the motion of each vehicle separately, the model operates with the average characteristics of the vehicles, and ignores any individual differences in power, inertia, braking distances, etc.

2. It is assumed that, on average, the response of all drivers to a change in the road conditions is adequate, namely, it is assumed that, on seeing a red traffic light or a speed limitation sign, for example, that there is a "sleeping policeman" ahead, or a pile-up of vehicles in front, the driver shows down to a complete stop or to a permissible speed, and does not continue to accelerate and subsequently have to use emergency braking.

3. It is assumed that all drivers obey the traffic rules, in particular, they do not exceed the maximum speed permitted on the road, and maintain a safe distance between the vehicles, depending on the speed.

The equation of the change in speed can then be written in the form

$$\begin{gathered}\frac{d\upsilon}{dt} = a; \quad a = \max\{-a^-, \min\{a^+, a'\}\} \\ a' = \sigma_0 a_\rho + (1-\sigma_0)\int_0^Y \omega(y) a_\rho(t, x+y)dy + \frac{V(\rho)-\upsilon}{\tau}, \quad a_\rho = -\frac{k^2}{\rho}\frac{\partial\rho}{\partial x}\end{gathered} \tag{1.2}$$

Here a is the acceleration of the traffic flow, a^+ is the maximum possible acceleration, a^- is the emergency braking deceleration, and the quantities a^+ and a^- are positive and are defined by the technical characteristics of each vehicle. The parameter $k > 0$ is, as has been shown previously [8–10], the propagation velocity of small perturbations ("the velocity of sound") in traffic flow. The parameter τ has the meaning of the delay time due to the finiteness of the speed of the driver's reaction to a change in the road conditions and the technical characteristics of his vehicle. This parameter corresponds to the tendency of the driver to maintain a speed corresponding to the maximum safe speed $V(\rho)$ for the flow density ρ [8, 10]

$$V(\rho) = \begin{cases} -k\ln\rho, & \upsilon < \upsilon^0_{\max} \\ \upsilon^0_{\max}, & \upsilon \geq \upsilon^0_{\max} \end{cases}$$

The speed $V(\rho)$ is determined from the condition for the car speed υ to depend on the flow density of ρ for the conditions of a simple wave, while occurs when the flow starts to spread out from the point where $\rho_0 = 1$ and $\upsilon = 0$, taking into account the limitation on the maximum permissible speed ($\upsilon \leq \upsilon^0_{\max}$). The value of the parameter τ may be different, depending on whether it is necessary to decelerate or accelerate in order to reach the maximum safe speed $V(\rho)$, namely

$$\tau = \begin{cases} \tau^+, & V(\rho) < \upsilon \\ \tau^-, & V(\rho) \geq \upsilon \end{cases}$$

The remaining parameters in formulae (1.2) have the following meaning: $Y = \min\{Y_0, L - x\}$ is the characteristic visibility along the flow, which depends on the weather conditions, $\omega(y)$ is the "weight" of the state of the flow in front of the vehicle for taking a decision on whether to change the type of driving, which can be defined, for example, as follows:

$$\omega(y) = \frac{\omega_0(y)}{\int_0^Y \omega_0(y)dy}, \quad \omega_0 = \begin{cases} 1, & 0 \le y \le Y_0 \\ 0, & y < 0,\ y > Y_0 \end{cases}$$

and σ_0 is a dimensionless parameter ($0 \le \sigma_0 \le 1$), characterizing the "weight" of the local situation compared with the situation at a certain distance in front of the vehicle.

Hence, in the expression for the acceleration of the traffic flow (1.2) the first term corresponds to the effect of the local situation, the second term corresponds to the effect of the situation in front at a distance less than or equal to the characteristic visibility Y, while the third corresponds to the adjustment in the speed of the vehicle to the maximum safe one for the actual flow density ρ.

An estimate of the value of the propagation velocity of small perturbations k was made previously in [8, 10], starting from the following considerations. Suppose, starting the motion from a state of rest ($\upsilon = 0$, $\rho = 1$) and accelerating to a velocity $\upsilon^0_{\max}$, the flow reaches a maximum permissible density, ρ_*, which guarantees that the motion is safe. We mean by a safe density that for which the distance between the vehicles is no less than the braking distance $X(\upsilon)$. Then,

$$\rho_* = (1 + X(\upsilon^0_{\max})/l)^{-1}, \quad k = \upsilon^0_{\max} \ln^{-1}(1 + X(\upsilon^0_{\max})/l)$$

When $\upsilon^0_{\max} = 80$ km/h the braking distance of a *VAZ* type care is 45 m, which, for a mean length of the car (taking into account the minimum distance between stopped cars) $l = 5$ m gives a propagation velocity of weak perturbations $k = 35$ km/h. For such a value of $\upsilon^0_{\max}$ the maximum possible safe flow density is $\rho_* = 0.1$. The maximum accelerations for cars of this class are $a^+ = 1.63$ m/s^2 and $a^- = 5.5$ m/s^2.

The "velocity of sound" k, estimated in this way, agrees well with experimental data [3, 4].

Hence, to describe the dynamics of traffic flow along a single-lane highway, from Eqs (1.1) and (1.2) we obtain a system of two quasi-linear partial differential equations in divergent form

$$\frac{\partial\rho}{\partial t} + \frac{\partial(\rho\upsilon)}{\partial x} = 0, \quad \frac{\partial(\rho\upsilon)}{\partial t} + \frac{\partial(\rho\upsilon^2)}{\partial x} = \rho a$$

The acceleration a is given by the last three formulae of (1.2).

We will formulate the boundary conditions at the ends of the part of the arterial road $0 \le x \le L$. Two versions of the boundary conditions are possible at the beginning of the flow where $x = 0$:

1. when there is no "jam", the flow density and the maximum safe speed for the given density is specified:

$$\rho(0, t) = \rho_0, \quad \upsilon(0, t) = V(\rho_0)$$

2. under conditions of a travelling or fixed "jam", adjoining the entrance part of the highway $x = 0$, we impose the condition that the density gradient is equal to zero, while the speed is equal to the maximum safe speed for the given density:

$$\partial\rho/\partial x|_{x=0} = 0, \quad \upsilon(0, t) = V(\rho)$$

The presence or absence of a "travelling jam", close to the left boundary of the calculated region ($x = 0$), is determined after calculating the next time step according to the following criterion: if

$$\partial\rho/\partial x|_{x=0} > 0 \quad \text{and} \quad \rho > \rho_0$$

then there is a "travelling jam".

At the exit of the flow, when $x = L$, we impose the "free exit" condition

$$\partial\rho/\partial x = 0, \quad \partial\upsilon/\partial x = 0$$

We will take as the initial conditions the fact that on a part of length x_0, measured from the entrance ($x = 0$), the arterial road is occupied by a flow of vehicles of density ρ_0, moving at a speed $V(\rho_0)$, and when $x_0 < x \le L$ the road is free of vehicles ($\rho = 0$, $\upsilon = 0$).

2. MODELS OF SYSTEMS FOR CONTROLLING ROAD TRAFFIC

We will consider two versions of the traffic control, characteristic for city roads: traffic lights and so-called "sleeping policemen".

Traffic lights. The main parameters of the operation of traffic lights are the duration of the signals: the green light t_g, the yellow light t_y and the red light t_r respectively. We propose the following algorithm to model the operation of traffic lights.

1. At the instant of switching from the green light to the yellow light we calculate the distance

$$x_r = (\upsilon^0_{\max})^2/(2a_r)$$

where a_r is the regular braking deceleration, which is less than the emergency braking value a^-. Vehicles which are a distance less than x_r from the traffic lights are unable to stop before the traffic lights with a standard braking deceleration of a_r, and hence they cross on the yellow light, which corresponds to the rules of traffic motion.

2. While the yellow light is operating we will assume that the maximum speed is

$$\upsilon^e_{\max}(x_l) = \upsilon^0_{\max} t_{ys}/t_y$$

where t_{ys} is the time which has elapsed since the yellow light showed. Then x_1 is a point which is shifted towards the traffic light in accordance with the relation

$$x_l = L_1 - x_r t_{ys}/t_y$$

where L_1 is the coordinate of the point where the traffic control system is situated (in this case, the traffic lights). As a result of this, at the instant when the red light shows vehicles stop at the traffic lights.

3. At the instant when the red light is switched to a green light the maximum permitted speed of crossing the traffic lights will be $\upsilon^0_{\max}$, as at the remaining points of the section of road considered.

"Sleeping policemen". The system of limiting the speed of traffic flow, usually called "sleeping policemen", is modelled by specifying that the maximum speed $\upsilon_{\max}$ at the point where the "sleeping policemen" is situated is considerably less than for the main part of the road – $\upsilon^0_{\max}$. In this paper we consider the case of two "sleeping policemen" at a distance d from the one another, which is the situation most often encountered in practice. The point where the first "sleeping policeman" is situated is $x = L_1$. Then the maximum permitted speed along the section of road considered $0 \le x \le L$ is specified as follows:

$$\upsilon_{\max} = \begin{cases} \upsilon_p, & x \in \{L_1, L_1 + d\} \\ \upsilon^0_{\max}, & x \in [0, L]/\{L_1, L_1 + d\} \end{cases}$$

where $\upsilon_p < \upsilon^0_{\max}$, and the parameter υ_p (the maximum crossing speed) of a "sleeping policeman" is one of the fundamental parameters of the model.

3. RESULTS OF NUMERICAL CALCULATIONS

The above problems were solved numerically by the TVD method with second order of accuracy [11]. The number of nodes in the calculation grid was 201.

We used the following parameters in the calculations: $L = 1000$ m is the length of the calculated region, $x_0 = 100$ m is the length of the section occupied by the moving traffic at the initial instant of time $t = 0$, $L_1 = 500$ m is the point where the traffic control systems are situated (the traffic lights or the first "sleeping policeman"), $d = 50$ m is the distance between two "sleeping policemen", $\rho_0 = 0.1 = 0.5$ is the traffic flow density at the entrance to the calculated region $x = 0$, $\upsilon^0_{\max} = 25$ m/s is the maximum speed on the main part of the road, $\upsilon_p = 3$ m/s is the maximum speed of crossing a "sleeping policeman", $k = 7.9$ m/s is the propagation velocity of small perturbations in the traffic flow, $a^+ = 1.5$ m/s^2 is the maximum acceleration of the flow, $a^- = 5$ m/s^2 is the maximum (emergency) braking deceleration of the flow, $a_r = 1.5$ m/s^2 is the standard braking deceleration, $Y_0 = 100$ m is the characteristic forward visibility along the flow, $\sigma_0 = 0.7$ is the "weight" of the local situation, $\tau^+ = 3.3$ s, $\tau^- = \infty$ is the time taken to adjust to a safe speed, and $t_g = 40 - 300$ s, $t_y = 5$ s and $t_r = 30$ s are the durations of the traffic-light signals.

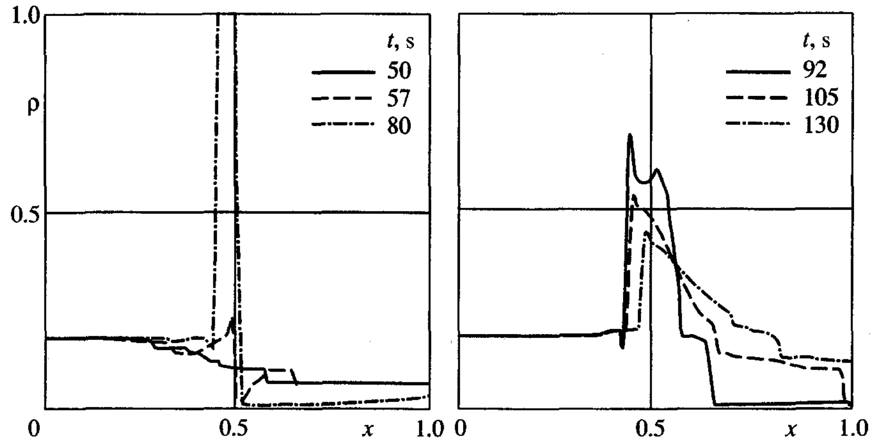

Fig. 1

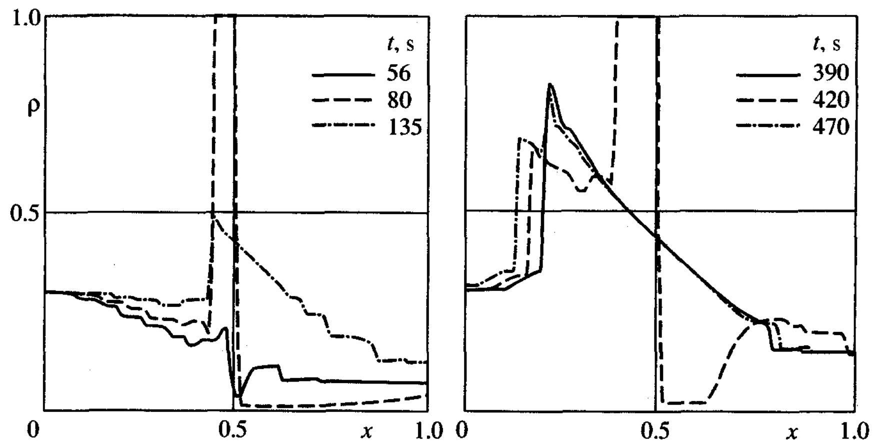

Fig. 2

Hence, in the calculations we varied the density of the incoming flow ρ_0 (and, of course, its speed) and the duration t_g for which the green light operates.

The results of the calculations are presented in Figs 1–4 and in the table.

In Figs 1 and 2 we show the distributions of the traffic flow density ρ with respect to the coordinate of the calculated region x at different instants of time, indicated on the figures, in the case when the flow is controlled by the traffic lights. In this case the working time of the green light t_g = 50 s, while the initial traffic flow densities are $\rho_0 = 0.18$ (Fig. 1) and $\rho_0 = 0.3$ (Fig. 2). The time in Fig. 1 corresponds to the following operating cycles and signals of the traffic lights: 50 s – the first cycle, end of the green light, 57 s – the first cycle, the yellow light, 80 s – the first cycle, the end of the red light, 92 s – the second cycle, the green light, 105 s – the second cycle, the green light, 130 s – the second cycle, and the end of the red light. In Fig. 2: 56 s – the first cycle, the yellow light, 86 s – the fifth cycle, the end of the red light, 135 s – the second cycle, the end of the green light, 390 s – the fif cycle, the end of the green light, 426 s – the fifth cycle, the end of the red light, 476 s – the sixth cycle, the end of the green light. As can be seen from these graphs, when $\rho_0 = 0.18$ a "travelling jam" is not formed (Fig. 1), while in the case when $\rho_0 = 0.3$ a travelling jam is formed, which moves in the opposite direction to the traffic flow, the speed of the vehicles in which is reduced considerably.

The results of an investigation of how the limiting initial flow density ρ_0^*, for which a travelling jam is not formed, depends on the duration of the green light t_g, are presented in the table. The remaining initial parameters are fixed. The dependence of ρ_0^* on t_g is described quite well by the following formula

$$\rho_0^* = a\ln(bt_g) \tag{3.1}$$

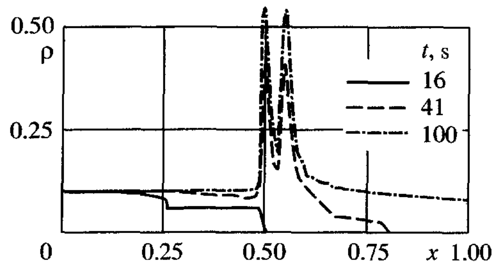

Fig. 3

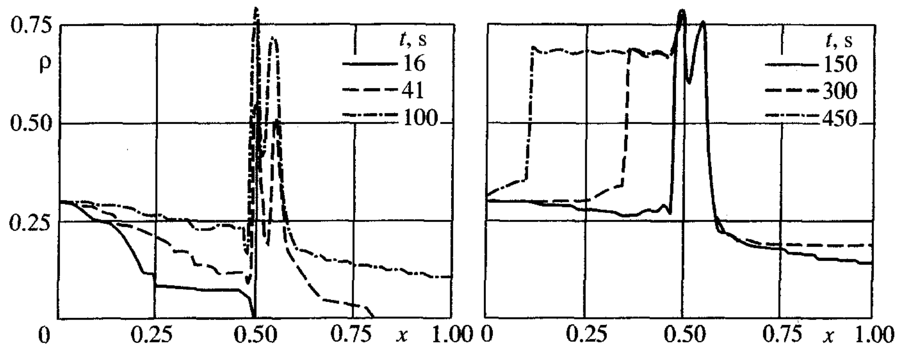

Fig. 4

where a and b are parameters which depend on many factors, including the duration of the red signal t_r. For the initial data considered $a = 0.054$ and $b = 0.87$. The difference Δ between the limiting densities ρ_0^*, calculated from formula (3.1), and the values obtained by numerical modelling, are also shown in the table. The root mean square deviation is equal to 0.0116, while the maximum difference in the densities $\Delta = 0.0216$.

t_g, s	40	60	80	100	150	200	250	300
ρ_0^*	0.18	0.21	0.23	0.23	0.27	0.29	0.31	0.31
$\Delta \times 10^3$	11.68	3.56	−0.89	11.16	−6.95	−11.41	−19.36	−9.52

In Figs 3 and 4 we show profiles of the traffic flow density ρ with respect to the coordinate of the calculated region x for the case when the flow is controlled by two "sleeping policemen" at different successive time intervals, indicated on the graphs, for an initial flow density $\rho_0 = 0.1$ (Fig. 3) and $\rho_0 = 0.3$ (Fig. 4).

The case when $\rho_0 = 0.1$ corresponds to free motion of the traffic flow through the zone in which the flow is controlled by the "sleeping policeman", while when $\rho_0 = 0.3$ a travelling jam is formed, which moves in the opposite direction to the flow. It can be seen that when the flow of vehicles traverses the section with the "sleeping policemen", two sections of increased density are formed (Figs 3 and 4), which, when $\rho_0 < 0.2$ does not impede the free passage of the flow through the obstacles. If $\rho_0 \geq 0.2$, there is a travelling jam before the "sleeping policemen", the occurrence of which leads to a situation in which, over the course of time, the density ρ at the entrance to the calculated region $x = 0$ begins to exceed the initial density ρ_0 and the motion before the obstacle zone becomes very slow (Fig. 4).

4. CONCLUSION

Our calculations enable us to conclude that, at low densities of the incoming traffic flow, "sleeping policemen" enable the speed to be controlled in the required way along the sections where they are installed, without interfering with the free motion of the traffic. However, when the density of the incoming traffic flow increases, they produce a "travelling jam", which moves in the opposite direction to the traffic flow, which, in the final analysis, leads to congestion on the road. Control of the traffic using traffic lights enables one, by choosing the optimum mode of operation (the duration of the signals of different colour), to increase the throughout considerably.

The model takes into account the main property of traffic flows, namely, self-organization, and enables the conditions required to ensure maximum throughput, the occurrence and evolution of "travelling jams" on roads, and the effect of the main components of traffic control, to be correctly described both qualitatively and quantitatively.

This research was supported financially by the Government of Moscow (GA-91/02) and by the Government of the Region of Brussels (RIB-2002//001).

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Translated by R.C.G.